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THE RENORMALIZATION-GROUP METHOD IN PROBLEMS OF MECHANICS†

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"Renormalization-group? It is very simple".
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The basic ideas of the renormalization-group (RG) method are presented. In particular, the concept of renormalization invariance, the results of which are the functional and differential RG equations is explained. Ways of solving the RG differential equation, as well as some technical aspects of the RG method, such as the method of ε-expansion, are outlined. The examples presented, which relate to the analysis of various mechanical problems, serve to illustrate the RG method and should promote its better understanding. © 2004 Elsevier Ltd. All rights reserved.

When constructing solutions of complex non-linear partial differential equations, symmetry arguments may turn-out to be very helpful. If the equations, as well as the initial and boundary conditions, are invariant under some group of transformations, the solution has to be constructed in terms of invariants of this group, namely, in terms of self-similar variables. The employment of self-similar variables enables one to reduce the order of the equation, or, sometimes, to transfer from partial differential equations to ordinary differential equations. At present the method of finding the symmetry groups of differential equations and constructing invariant solutions by changing to self-similar variables, the foundation of which were laid by Sofus Lie, is well developed [1, 2].

However, a certain additional group of symmetry transformations exists that is not directly connected with the form of the differential equations, but follows from the arbitrariness in setting the initial or boundary conditions of the problem. The variety of transformations, which leave the solution unchanged under a transformation describing the transfer from one method of setting the initial or boundary conditions to another, forms a symmetry group called a renormalization-group (RG) or, in abbreviated form, the renormgroup, and the use of an RG to construct solutions of differential equations is called the RG method. In many cases the application of the RG method gives trivial results. However, in the case of a non-linear multimode system, when modes of all scales are equally essential for understanding the phenomena which occur, the methods of classical mechanics and mathematical physics, where a finite number of interacting modes is commonly considered, turn out to be helpless. In this case the RG method provides a powerful tool for describing such systems. In some cases the equation that describes one or another physical process may turn out to be unknown, and the requirement of RG invariance may be able to replace this deficient equation.

Initially the RG method originated in quantum field theory, and the property of RG invariance appeared to be connected with an ambiguity in the procedure of renormalization used to remove divergences within the framework of perturbation theory [3]. This method was then developed further by Wilson and was successfully used to describe critical phenomena in phase transitions of the second kind [4, 5].

The aim of this paper is to outline in the simplest form the essence and specific features of the method by simple and obvious examples, without referring to quantum field theory and the theory of critical phenomena, which will be understandable by a wide range of researchers who work in the field of applied mathematics and mechanics. In view of the nature of this review, references are commonly given to surveys and textbooks rather than to original papers. The examples describing various mechanical processes are of an illustrative character and are to a large extent based on publications by the author.

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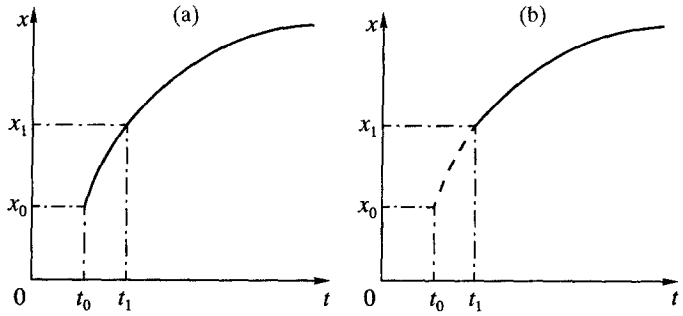

Fig. 1

1. RENORMALIZATION-GROUP INVARIANCE

For an initial understanding of the concept of renormgroup invariance (RG invariance) we will consider several simple examples that we shall call trivial. The meaning of the term "trivial" will be explained a little later.

1. Suppose we are given some differential equation

$$dx/dt = V(x) \tag{1.1}$$

that can be interpreted as an equation of the trajectories of a point mass moving in a given steady field of velocities.

The unique solution of this equation in defined by two numerical parameters, namely, the given initial instant of time t_0 and the given initial value of the coordinate x_0 (Fig. 1a). Due to the invariance of the equation under a time shift, the solution will depend only on the time difference

$$x(t) = X(t - t_0; x_0) \tag{1.2}$$

and the function $X(t - t_0; x_0)$ obeys the relation

$$X(0; x_0) = x_0 \tag{1.3}$$

that follows from the initial condition.

If we choose as the initial instant of time a certain instant t_1 and as the initial value we choose a point on the trajectory that corresponds to this instant of time (see Fig. 1b), that is

$$x_1 = X(t_1 - t_0; x_0) \tag{1.4}$$

the form of the trajectory when $t \geqslant t_1$ will not change, and the solution can be represented in the form

$$x(t) = X(t - t_1; x_1) = X(t - t_0; x_0) \tag{1.5}$$

Thus, the solution of Eq. (1.1) must obey some functional relation that follows from the property that the trajectory of motion is independent of the choice of the initial point on the trajectory, namely, of the way of specifying the initial conditions. According to Eqs (1.4) and (1.5) this relation, when $t_0 = 0$ and $t_1 = \tau$, has the form

$$X(t; x_0) = X(t - \tau; X(\tau; x_0)) \tag{1.6}$$

If we consider the second-order equation of motion

$$d^2x/dt^2 = F(x) \tag{1.7}$$

the solution of this equation will depend on three numerical parameters x_0, $v_0 = dx(t)/dt|_{t=t_0}$ and t_0, which specify the initial conditions of the problem, i.e.

$$x(t) = X(t - t_0; x_0, v_0) \tag{1.8}$$

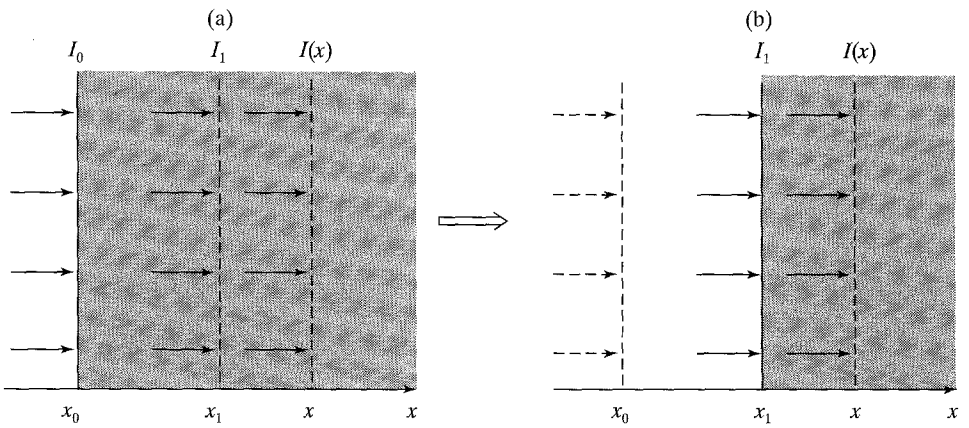

Fig. 2

and, correspondingly, for the velocity we will have

$$v(t) = dx/dt = V(t-t_0; x_0, v_0) \tag{1.9}$$

In this case

$$X(0; x_0, v_0) = x_0, \quad V(0; x_0, v_0) = v_0 \tag{1.10}$$

The fact that the form of the trajectory is independent of the way the initial conditions are specified (arbitrariness in the choice of the starting point on the trajectory) leads, in the case of the second-order equation, to two functional equations

$$\begin{aligned} x(t) &= X(t; x_0, v_0) = X(t-\tau; X(\tau; x_0, v_0), V(\tau; x_0, v_0)) \\ v(t) &= V(t; x_0, v_0) = V(t-\tau; X(\tau; x_0, v_0), V(\tau; x_0, v_0)) \end{aligned} \tag{1.11}$$

which satisfy conditions (1.10).

Note that functional equations (1.6) and (1.11) are independent of the form of the function $V(x)$ in Eq. (1.1) and of $F(x)$ in Eq. (1.7), that is, they express the properties of solutions of equations that belong to a certain specific class.

2. As the second trivial example we will consider the plane problem of radiation transport in a uniform absorbing medium [6]. Suppose radiation of intensity I_0 is incident on a boundary surface having coordinates x_0 (see Fig. 2a). When the radiation propagates in the medium, its intensity varies in some way

$$I(x) = f(x-x_0; I_0) \tag{1.12}$$

and at the point x_1 the intensity will have the value

$$I_1 = f(x_1-x_0; I_0) \tag{1.13}$$

According to Ambartsumyan's principle, when $x > x_1$ one can assume that radiation of intensity I_1 is incident on the boundary at the point x, (Fig. 2b). In this case the intensity at the point $x > x_1$ will be defined by the relation

$$f(x-x_0; I_0) = f(x-x_1; I_1) \tag{1.14}$$

Application of relations (1.13) and (1.14) leads to the fact that the function f must obey the functional equation

$$f(x-x_0; I_0) = f(x-x_1; f(x_1-x_0; I_0)) \tag{1.15}$$

which is identical with Eq. (1.6) obtained above.

3. One more example, which illustrates the property of RG invariance, is the solution of the initial Cauchy problem for the diffusion equation

$$[\partial_t - \kappa\Delta]C(\mathbf{r}, t) = 0, \quad C(\mathbf{r}, t_0) = \varphi_0(\mathbf{r}) \tag{1.16}$$

The solution of this equation can be represented in the form of the Duhamel integral

$$C(\mathbf{r}, t) = \int D(\mathbf{r} - \mathbf{r}_0, t - t_0)\varphi_0(\mathbf{r}_0)d\mathbf{r}_0 \tag{1.17}$$

where the function $D(\mathbf{r}, t)$ is the solution of the diffusion equation (1.16) obeying the initial condition $D(\mathbf{r}, 0) = \delta(\mathbf{r})$.

This function is connected with Green's function of the diffusion equation by the relation $G(\mathbf{r}, t) = H(t)D(\mathbf{r}, t)$ (here $H(t)$ is Heaviside's function).

At the instant t_1 we have

$$C(\mathbf{r}, t_1) = \int D(\mathbf{r} - \mathbf{r}_0, t_1 - t_0)\varphi(\mathbf{r}_0)d\mathbf{r}_0 = \varphi_1(\mathbf{r}) \tag{1.18}$$

When $t > t_1$ one can consider the Cauchy problem for the initial instant t_1 with initial condition (1.18).

Again writing the solution in the form of the Duhamel integral and taking into account relation (1.18) we obtain

$$\begin{aligned} C(\mathbf{r}, t) &= \int D(\mathbf{r} - \mathbf{r}_1, t - t_1)\varphi_1(\mathbf{r}_1)d\mathbf{r}_1 = \\ &= \int D(\mathbf{r} - \mathbf{r}_1, t - t_1)D(\mathbf{r}_1 - \mathbf{r}_0, t_1 - t_0)\varphi_0(\mathbf{r}_0)d\mathbf{r}_1 d\mathbf{r}_0, \quad t > t_1 > t_0 \end{aligned} \tag{1.19}$$

Comparing expressions (1.16) and (1.19) we obtain

$$D(\mathbf{r} - \mathbf{r}_0, t - t_0) = \int D(\mathbf{r} - \mathbf{r}_1, t - t_1)D(\mathbf{r}_1 - \mathbf{r}_0, t_1 - t_0)d\mathbf{r}_1, \quad t > t_1 > t_0 \tag{1.20}$$

In the d-dimensional case, solving Eq. (1.16) gives

$$D(\mathbf{r} - \mathbf{r}_0, t - t_0) = \frac{1}{\pi^{d/2}R(t - t_0)^d}\exp\left\{-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{R(t)}\right\}, \quad R^2(t) = 4\kappa(t - t_0) \tag{1.21}$$

It can be shown by a direct calculation (see, for example, [7]) that the function (1.21) obeys relation (1.20), called the semigroup property.

The function $D(\mathbf{r} - \mathbf{r}_0, t - t_0)$ can be interpreted as the probability of finding a Brownian particle at the point $\mathbf{r}$ at the instant t if at the initial instant t_0 the particle was situated at the point $\mathbf{r}_0$ (this is so called the conditional or transient probability). With this interpretation relation (1.20) is nothing more than the Einstein–Smoluckhowski–Kolmogorov–Champman equation [8]. This equation reflects the Markov character of the process of the random walk of a Brownian particle, which consists of the fact that the behaviour of a random Markov process after the instant t, with a given distribution at the instant t, is independent of its previous behaviour (before the instant t).

Note that, in fact, when deriving relation (1.20) the diffusion equation was not used in explicit form; only the possibility of writing the solution in the form of the Duhamel integral was used. By employing standard methods [8], from Eq. (1.20) one can obtain two differential equations for the function $D(\mathbf{r} - \mathbf{r}_0, t - t_0)$, known as the direct and inverse Kolmogorov equation; the diffusion equation is a special case of these equations.

In all examples considered above we were dealing with the property that the functional form of the solution is independent of the method of specifying the initial or boundary conditions. This property has been given the name of functional self-similarity [9], which is a generalization of scale similarity and the method of dimensional analysis connected with it.

Unlike the property of scale similarity, which consists in preserving the form of the solution under a coordinate extension (a change in scale), in the case of functional self-similarity the form of the solution remains unchanged under coordinate extension (or a shift in the origin of coordinates) and a corresponding change (renormalization) in some numerical parameters of the problem. In this context, for example, all ellipses are self-similar since they are distinguished by the value of the scale (the length of the principal semiaxis) and a numerical parameter (the reccentricity).

In the cases considered above the condition of functional self-similarity consists of the invariability (invariance) of the form of the solution under a transformation (shift) of the argument t or x and a change in some numerical parameter g (such parameters were x_0 and I_0 in Example 1 and 2, whereas in Example 3 the width of the initial distribution $R(t_0)$ served as the parameter).

The operation of combined transformation of the coordinates (shift or extension) and renormalization of a numerical parameter, under which the form of the solution is preserved, can be represented as

$$t \Rightarrow t_1 = t - \tau, \quad g \Rightarrow g_1 = f(\tau, g) \tag{1.22}$$

Transformations (1.22) obey the group composition law, according to which successive execution of two transformations with group parameters τ_1 and τ_2 is a transformation of the same form with the parameter $\tau_1 + \tau_2$. This law follows from functional equation (1.6)

$$g_2 = f(\tau_1 + \tau_2, g) = f(\tau_2, f(\tau_1, g)) \equiv f(\tau_2, g_1) \tag{1.23}$$

The set of transformation (1.22) forms a one-parameter continuous group, in which the transformation with group parameter $\tau = 0$ corresponds to the identical group element, whereas the group element with the parameter $-\tau$ corresponds to the inverse one. A group of transformations that includes a shift of the independent variable and an appropriate change (renormalization) in the numerical parameter or a system of parameters form a renormalization-group. For a wide class of models of physical systems the equations that describe their behaviour possess invariance under RG transformations [9, 10].

More frequently a formulation with a multiplicative law of the group parameter transformation rather than with an additive one is used. To change to the multiplicative formulation we will make the change of variables $t = \ln x$, $\tau = \ln\lambda$ and introduce the notation $f(\ln x, g) = F(x, g)$. Then Eq. (1.6) will become

$$F(x, g) = F(x/\lambda, F(\lambda, g)) \tag{1.24}$$

In the case of the multiplicative version the set of RG transformations will include the operation of extension (a change of scale) and a transformation of the numerical parameter

$$x \Rightarrow x' = x/\lambda, \quad g \Rightarrow g' = F(\lambda, g), \quad F(1, g) = g \tag{1.25}$$

An extension of transformation (1.25) to the case when the problem contains a certain additional parameter y with the dimension of length is the transformation

$$x \Rightarrow x' = x/\lambda, \quad y \Rightarrow y' = y/\lambda, \quad g \Rightarrow g' = F(\lambda, y, g) \tag{1.26}$$

In this case the function $F(x, y, g)$ satisfies the functional equation

$$F(x, y, g) = F(x/\lambda, y/\lambda, F(\lambda, y, g)), \quad F(1, y, g) = g \tag{1.27}$$

From this equation one can find a differential equation for the function $F(x, y, g)$. If we differentiate Eq. (1.27) with respect to λ and then put $\lambda = 1$, we arrive at the RG differential equation (Ovsyannikov's compensational equation [3], which is known in the western literature as the Callan–Symanzic equation)

$$\left\{-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \beta(y, g)\frac{\partial}{\partial g}\right\}F(x, y, g) = 0 \tag{1.28}$$

where the function $\beta(y, g)$ (referred to as the RG-function, the Gell–Mann–Low function [11] and the Wilson function [4]) is defined by the relation

$$\beta(y, g) = \partial F(x, y, g)/\partial x|_{x=1} \tag{1.29}$$

In the case of the additive version, the differential RG equation follows from Eq. (1.6) and has the form

$$\left\{-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right\}f(t, g) = 0; \quad \beta(g) = \partial f(t, g)/\partial t|_{t=0} \tag{1.30}$$

If we differentiate Eq. (1.6) with respect to τ and then put $\tau = t$, then, taking into account relation (1.3), we obtain an equation of another kind, namely, the so-called evolutionary equation of Bogolyubov and Shirkov

$$\partial f(\tau, x_0)/\partial\tau = \beta[f(\tau, x_0)] \tag{1.31}$$

which can be treated as the equation of motion of an inertialess particle with the coordinate $X(\tau) = f(\tau, x_0)$ in a given non-uniform steady flow $\beta(X)$ (see Eq. (1.1)).

2. DISCUSSION OF THE EXAMPLES CONSIDERED ABOVE

For a given function $\beta(g)$, solution of differential equation (1.30) enables one to obtain the form of the function $f(t, g)$. The solution is found by the method of characteristics, and the corresponding equation for the characteristic has the form

$$-\frac{dt}{1} = \frac{dg}{\beta(g)} \tag{2.1}$$

If we put $g = x$, $\beta = -V(x)$, we arrive at the original differential equation of problem (1.1). As the result, in Example 1 the use of the arguments of RG invariance (functional self-similarity) does not give any new information about the system, and in this context Example 1 and 3 are trivial. In Example 2 the equation for the intensity was not considered at all, and only Ambartsumyan's principle was used. In this case the equation for the characteristic (2.1) shows what this unknown equation can be – it turns out that it must be a first-order differential equation of of the form

$$dI/dx = -\beta(I)$$

In fact, in Example 3 the form of the equation for the density was also not used when obtaining functional equation (1.20), but only the Einstein–Smoluchowski–Kolmogorov–Chapman relation, which reflects the Markovran nature of the diffusion random process, was employed. In this problem the equation for the characteristic of the differential RG equation corresponds to the inverse Kolmogorov equation of the theory of random process. A common feature of all the examples considered is the fact that in these equations the evolution with respect to the temporal or spatial variable is described by a *differential equation*.

A use of a differential equation means that the temporal (or spatial) derivative is specified by the state at a given instant of time (or at a given point in space) and is dependent of the state of the system at preceding instants of time (independent of the previous history of the process). Such evolutionary processes are referred to as Markovran process. In a Markovran process the evolution over a finite interval can be treated as a sequence of evolutions at all intermediate stages, and the solution can be expressed in the form of an integral over time from the initial instant to the final one. In a similar way, when analysing "spatial evolution" (Example 2) a transition from one point to another proceeds as a consequence of transitions from a given point to the nearest neighbouring point through all intermediate points. Such processes can be conditionally called Markovran process in a generalized sense. An example of such a process may be the consideration of the evolution of a wave front using the Huygens–Fresnel principle in the theory of wave propagation in a space of odd dimensionality or a description of the cascade process of energy transport along the wave-number spectrum in the theory of developed turbulence.

3. NON-TRIVIAL EXAMPLES OF RENORM-GROUP INVARIANCE

In the examples presented above the equations of the characteristics of the RG differential equation turned out to be identical with the original equations of the problem, and in this context these examples were said to be trivial. If the RG equation or the equation of the characteristics are not identical with the original equations of the system or these equations are generally unknown, we have non-trivial cases. These cases are mainly connected with an investigation of evolution in the space of scales.

An example is the investigation of the effect of screening on the effective charge in a dielectric medium. This approach was used for the first time in quantum electrodynamics, where the physical vacuum plays the role of the dielectric medium [11].

If we put into a dielectric medium a charge q_0, distributed inside a sphere of radius a_0, a screening charge arises due to polarization effects. As a result, the charge q inside a sphere of radius $r > a_0$ will

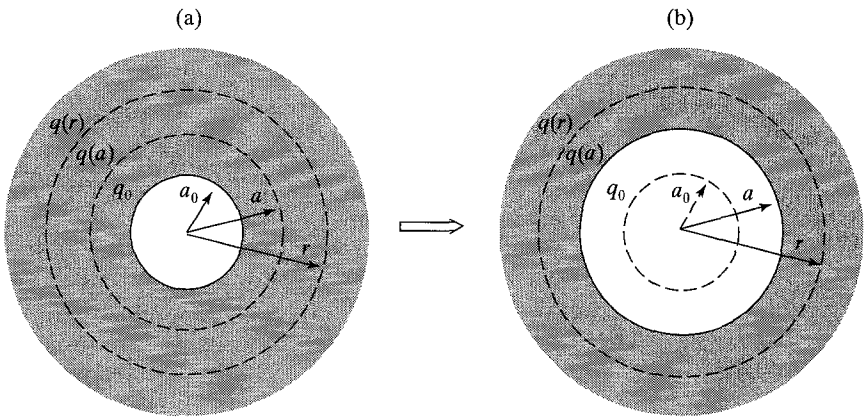

Fig. 3

be equal to the sum of the original charge q_0 and a screening charge of opposite sign, that is, $q(r) = q_0 + \delta q$ (Fig. 3a). Hence, the total effective charge will be a certain function of the value of the original charge q_0, the radius of the original charge a_0 and other characteristic parameters of the problem L, with the dimension of length

$$q(r) = f(r; L, a_0, q_0), \quad f(a_0; L, a_0, q_0) = q_0 \tag{3.1}$$

According to formula (3.1), the total charge inside a sphere of radius $a > a_0$ is given by the relation

$$q(a) = f(a; L, a_0, q_0)$$

However, if we take a charge $q(a)$ (a renormalized charge) located inside a sphere of radius a and place it in a dielectric medium (see Fig. 3b), in the domain $r > a$ this charge will create a field identical to the field created by a charge q_0 located inside a sphere of radius a_0, and hence we must have the following relation for the resulting charge $q(r)$

$$q(r) = f(r; L, a_0, q_0) = f(r; L, a, q(a)) = f(r; L, a, f(a; L, a_0, q_0)) \tag{3.2}$$

Using the dimensionality arguments, we can write $q(r)$ in the form

$$q(r) = q_0\varphi\left(\frac{r}{a_0}, \frac{L}{a_0}, \frac{q(a_0)}{q_0}\right) = q_0\varphi\left(\frac{r}{a}, \frac{L}{a}, \frac{q(a)}{q_0}\right) \tag{3.3}$$

From the relation $q(a) = q_0\varphi(r/a, L/a, q(a)/q_0)|_{r=a}$ it follows that the function $\varphi(x, y, g)$ must obey "the normalization condition" $\varphi(1, y, g) = g$.

As a result, it turns out that the function $\varphi(x, y, g)$ must satisfy the RG functional equation

$$\varphi(x, y, g) = \varphi(x/\lambda, y/\lambda, \varphi(\lambda, y, g)) \tag{3.4}$$

which is identical with Eqs (1.6) and (1.15).

In this case the functional self-similarity expresses the invariability of the form of the function which specifies the dependence of the value of the screened charge on the choice of the position of the boundary (the radius of the sphere) with appropriate specification of the value of the function on the boundary (the renormalized charge inside the sphere).

If we choose an arbitrary radius a in functional equation (3.4) as the scale unit, we will have an equation that describes the evolution of the screened (effective) charge in the space of scales. Equation (3.4) and the RG differential equation following from it are not identical with the original system of equations of macroscopic electrodynamics, and in this sense the example considered is non-trivial.

If, on the basis of the dimensionality arguments, we take $dq/dr = -q/L$, the solution of the RG equation (see below) will have the form

$$q(r) = q_0\exp(-(r-a_0)/L) = q(a)\exp(-(r-a)/L)$$

It is obvious that the conditions of RG invariance (3.2) are satisfied.

The second example concerns the problem of subgrid modelling in turbulence, theory. According to the modern viewpoint [12] a state of developed turbulence, realized at large Reynolds numbers, is characterized by the excitation of a very large number (of the order of $Re^{9/4}$) of modes of different scales. Under steady conditions the energy input to the system is achieved in the region of large scales, this energy is transferred by a cascade sequence of interactions of modes of the nearest scales to the region of progressively smaller scales (energy transport along the wave-number spectrum), and the energy then dissipates in the small-scale region as a result of molecular viscosity. For a direct numerical simulation of the developed turbulence, the power of modern computers is insufficient to describe all scales that specify the turbulent regime, and hence, a part of scales turns out to be beyond the limits of resolution of the finite difference grid, that is, the small-scale (subgrid) modes turn out to be unaccounted for. However, the subgrid modes cannot simply be removed because they determine the sink of turbulent energy, which ensures the existence of the steady non-equilibrium regime. Thus, the problem arises of a direct numerical simulation of the large-scale (supergrid) modes, taking into the account the average effect of subgrid modes (the use of cells of finite size corresponds to averaging over the cell volume). This averaged effect is simulated by introducing a subgrid viscosity, which depends on the cell scale, similar to the way the average effect of molecular motions is taken into account by means of empirically defined transport coefficients. The form of the dependence of the subgrid viscosity on the cell scale $a = 2\pi/\Lambda$ is unknown, but it can be found from considerations of RG invariance [13].

In the above problem of the numerical simulation of large-scale processes the property of RG invariance consists of the natural requirement that the behaviour of the solution in the large-scale region should be independent of the grid scale with an appropriate choice of the dependence of the subgrid viscosity on the scale. If $F(k, \omega, L; \nu, \Lambda)$ is some function, which describes the system behaviour, the following relation must be satisfied

$$\begin{aligned} &F(k, \omega, L; \nu, \Lambda) = F(k, \omega, L; \nu_1, \Lambda_1) \\ &\nu_1 = f(\nu, L, \Lambda, \Lambda_1), \quad f(\nu, L, \Lambda, \Lambda) = \nu \end{aligned} \tag{3.5}$$

where L is a set of the characteristic numerical parameters of the problem (in Kolmogorov's turbulence theory the only parameter in the inertial range of the spectrum is the rate of energy dissipation). Differentiating equality (3.5) with respect to Λ_1 and then putting $\Lambda_1 = \Lambda$, we obtain the RG differential equation

$$\begin{aligned} &dF/d\Lambda = [\partial/\partial\Lambda + \beta(\nu, L, \Lambda)\partial/\partial\nu]F(k, \omega, L; \nu, \Lambda) = 0 \\ &\beta(\nu, L, \Lambda) = \partial f(\nu, L, \Lambda, \Lambda_1)/\partial\Lambda_1|_{\Lambda = \Lambda_1} \end{aligned} \tag{3.6}$$

Hence it follows that the solution of Eq. (3.6) must lie on the characteristic which satisfies the equation

$$d\nu/d\Lambda = \beta(\nu, L, \Lambda) \tag{3.7}$$

The function $\beta(\nu, L, \Lambda)$ can be found by calculating the contribution to the subgrid viscosity $\delta\nu$ of modes from a narrow wave-number range $\delta\Lambda$. To calculate this quantity it is sufficient to take into account a small number of modes. A knowledge of the function $\beta(\nu, L, \Lambda)$ enables one to find the dependence of the subgrid viscosity on the cell scale by solving the equation for the characteristic.

4. SOLUTION OF THE RENORM-GROUP EQUATIONS

To find a solution of the RG equation (1.32) it is necessary to know the RG-function $\beta(y, g)$. According to relation (1.33) the form of this function is defined by the behaviour of the solution in the vicinity of the point $x = 1$ (the normalization point). Thus, with the help of the requirement of RG invariance and knowing the behaviour of the function in a restricted domain one can find its behaviour over the whole domain in which the function is defined. Here it is pertinent to draw an analogy with the Lie theory of continuous groups when a knowledge of the operators of infinitesimal transformations (the group generators) enables one to construct operators of finite transformations using the requirement that the group composition law must be satisfied. In particular, this statement can be illustrated by the example of the group of translational transformation operators

$$\hat{T}(a)f(x) = f(x+a), \quad \hat{T}(0) = \hat{I} \tag{4.1}$$

The operator of infinitesimal translations can be easily constructed

$$\hat{T}(\delta a)f(x) = f(x+\delta a) = f(x)+\delta a df(x)/dx = [\hat{I}+\delta a\hat{p}]f(x) \tag{4.2}$$

where $\hat{p} = d/dx$ is the infinitesimal operator (the generator of the group of translations).

From the group composition law it follows that

$$\hat{T}(a+a_1) = \hat{T}(a)\hat{T}(a_1) \tag{4.3}$$

Putting $a_1 = \delta a$ and using equality (4.2), we obtain the equation

$$\delta\hat{T}(a)/\delta a = \hat{p}\hat{T}(a) \tag{4.4}$$

whose solution has the form

$$\hat{T}(a) = \exp(a\hat{p}) \tag{4.5}$$

If we assume $a = \sum_i \delta a_i$, then

$$\hat{T}(a) = \prod_i \hat{T}(\delta a_i) \tag{4.6}$$

Hence, a knowledge of the operator of infinitesimal transformation enables us to find the operator of finite transformations as a sequence of infinitesimal transformations.

A similar consideration of the operator of coordinate extension (scale transformation) defined by the relation

$$\hat{L}(\lambda)f(x) = f(\lambda x), \quad \hat{L}(1) = \hat{I}$$

using the group property $\hat{L}(\lambda)\cdot\hat{L}(\lambda_1) = \hat{L}(\lambda\lambda_1)$ and the form of the infinitesimal transformation operator $\hat{p} = xd/dx$ gives

$$\hat{L}(\lambda) = \lambda^{xd/dx}$$

In the case of the RG transformations (1.25) the RG-function defined, according to relation (1.33) by the system behaviour near the normalization point $x = 1$ plays the role of infinitesimal operator, and the system behaviour over the whole domain of definition is governed by a sequence of operations of infinitesimal transformations, which is found by solving the RG differential equation (1.33).

A general solution of the RG equation was found by Ovsyanikov in 1956 [14]. If there is no dependence on y (there is no additional characteristic scale), the function $F(x, g)$ satisfies the equation

$$\left\{-x\frac{\partial}{\partial x}+\beta(g)\frac{\partial}{\partial g}\right\}F(x,g) = 0 \tag{4.7}$$

and the boundary condition $F(1, g) = g$. In this case a solution in implicit form is given by the Gell-Mann–Low formula [11]

$$\int_g^{F(x,g)} \frac{dg'}{\beta(g')} = \ln x \tag{4.8}$$

The main contribution to integral (4.8) is made by the domain where the β-function vanishes. The point g^*, defined by the condition $\beta(g^*) = 0$, is called a fixed (stationary) point of the RG transformation. Near the fixed point the β-function can be represented in the form

$$\beta(g) = A(g-g^*) \tag{4.9}$$

and for the solution of Eq. (4.7) we will have

$$F(x,g) \approx g^* + (g-g^*)x^A \tag{4.10}$$

Hence it follows that, if $A = \partial\beta(g)/\partial g_{g=g^*} > 0$, the fixed point will be stable as $x \to 0$ (the infrared limit in the case when x is the wave number), whereas if $A < 0$, it will be stable as $x \to \infty$ (the ultraviolet limit).

Thus, a search for the asymptotics of the solution of the RG equation reduces to finding the roots of the equation $\beta(g) = 0$, which define the position of the fixed points, and determining the signs of the derivatives at these points, which specifies their stability in the infrared or ultraviolet limit. Since the asymptotic form does not depend on the position of the initial point (provided it lies in the domain of attraction of the fixed point), there is universality, namely, the behaviour of the system in the large-scale (or small-scale) domain is independent of the properties of the system in the domain of small (or large) scales. This universality is well-known in the theory of critical phenomena, for polymer systems, in the theory of developed turbulence, etc.

Among the fixed points a trivial one can exist, at which the actual parameter of expansion in series of perturbation theory (invariant charge) vanishes and there are no non-linear intermode interactions (asymptotic freedom occurs). The non-trivial fixed point describes the asymptotic behaviour of the system when there are intermode interactions.

We will now present two particular examples of the solution of RG equation (4.7).

1. Suppose that, in the vicinity of the point $x = 1$, the function $F(x, g)$ is linear in x and g

$$F(x, g) \approx g + \alpha g(x - 1) + \ldots \tag{4.11}$$

Then $\beta(g) = \alpha g$ and by equality (4.8) we obtain

$$F(x, g) = gx^{\alpha} \tag{4.12}$$

Hence, under the requirement of RG invariance the solution, which is linear in the vicinity of the point $x = 1$, turns out to be a power function, singular either at the point $x = 0$ when $\alpha < 0$ or at the point $x = \infty$ when $\alpha > 0$. In this case it turns out that formula (4.11) specifies the zeroth and first terms of the expansion of the power function in series in powers of $\ln x$ near the point $x = 1$.

$$F(x, g) = gx^{\alpha} = ge^{\alpha \ln x} \approx g[1 + \alpha \ln x + \ldots] \approx g + g\alpha(x - 1) + \ldots \tag{4.13}$$

2. Near the point $x = 1$ the function $F(x, g)$ is linear in x and quadratic in g

$$F(x, g) \approx g + Ag^2(x - 1) + \ldots \tag{4.14}$$

After calculating the function $\beta(g)$ and substituting it into equality (4.8) we obtain

$$F(x, g) = \frac{g}{1 - Ag \ln x} \tag{4.15}$$

Thus, it turns out that expression (4.14) is the zeroth and first terms of a geometric progression. The "true" solution, which satisfies the requirement of RG invariance, has a singularity at the point $x = \exp(1/Ag)$ and decreases as $x \to 0$ and as $x \to \infty$. Such behaviour would have been difficult to presume from a knowledge of the form of the function $F(x, g)$ in the vicinity of the point $x = 1$.

If an additional dimensional parameter L exists the required function will contain an additional independent variable y, and one must seek a solution of Eq. (1.32), which is also found by the method of characteristics [3]. In implicit form this solution is defined by the relation

$$\Phi(y, g) = \Phi(y/x, F(x, y, g)) \tag{4.16}$$

where the function $\Phi(y, g)$ is a solution of the equation of characteristic

$$-\frac{dy}{y} = \frac{dg}{\beta(g, y)} \tag{4.17}$$

written in the form

$$\Phi(y, g) = \text{const} \tag{4.18}$$

As an example consider the case $\beta = -g/y$. Then, written in the implicit form (4.18), the solution of the equation for the characteristic is given by the relation $\Phi(y, g) = g \exp(1/y)$. According to relation (4.16) we find

$$F(x, y, g) = g\exp(-(x-1)/y) = g_0\exp(-x/y) \tag{4.19}$$

This example corresponds to the description of charge screening due to polarization effects considered in Section 3. In this case, as the characteristic scale of the length one can use the Compton wavelength of an electron when considering the effect of the polarization of a vacuum in quantum electrodynamics or the Debye–Hückel screening radius in plasma theory. In this case $q(r) = q_0 \exp(-r/L)$ will correspond to solution (4.19).

When considering various problems by the RG method one also encounters functional equations of another form

$$f(x, y, g) = f(t, y, g)f(x/t, y/t, F(t, y, g)), \quad f(1, y, g) = 1 \tag{4.20}$$

where $F(x, y, g)$ is the solution of Eq. (1.32) (this is the so-called equation of the second class in the terminology of [3]).

Differentiating Eq. (4.20) with respect to t and then putting $t = 1$ we find the RG differential equation for the function f

$$\left\{-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \beta(y, g)\frac{\partial}{\partial g}\right\}f(x, y, g) = -\gamma(y, g)f(x, y, g)$$
$$\beta(y, g) = \partial F(x, y, g)/\partial x|_{x=1}, \quad \gamma(y, g) = \partial f(x, y, g)/\partial x|_{x=1} \tag{4.21}$$

The solution of Eq. (4.21) is given by the relation [3]

$$f(x, y, g) = \exp\{\Phi(y/x, F(x, y, g)) - \Phi(y, g)\} \tag{4.22}$$

where $\Phi(y, g)$ is the solution of the equation

$$\left\{-y\frac{\partial}{\partial y} + \beta(y, g)\frac{\partial}{\partial g}\right\}\Phi(y, g) = -\gamma(y, g) \tag{4.23}$$

5. THE RENORMALIZATION GROUP METHOD IN A FIELD THEORY FORMULATION

The concept of the renormalization group originally appeared in quantum field theory when considering a system of quantum interacting fields. A description of this system within the framework of perturbation theory, when non-interacting fields are considered as the unperturbed system, leads to integrals over momenta (wave numbers) of intermediate states. If the rate of decrease of the integrands is greater than the rate of increase of the integral measure (whose power of increase is defined by the space dimensionality), the integrals converge. In the opposite case divergences of the integrals arise at large momenta (ultraviolet divergences). In the intermediate case, when the rates of increase of the integral measure and of decrease of the integrands are equal, logarithmic divergences occur. In so-called renormalized theories it appears to be possible to treat the divergent expressions as additions to the original parameters of the theory (particle masses and coupling constants).

The idea of removing the divergences by means of renormalization consists of the proposal to treat the observable (physical) parameters of the theory as the sum of the original ("bare") parameter and a field addition to this parameter. If there are divergences of the field addition, to obtain a finite value of the physical parameter the value of the bare parameter (unobservable) has to be infinite. When formulating the theory in terms of observable physical quantities (the renormalized theory) a transition from bare parameters to observable (renormalized) parameters is performed. In the example of charge screening considered above the screened charge will be the physical parameter, while q_0 will be the bare one; in this case the renormalization constant $q_0/q(a)$ is finite. However, if there are divergences, the renormalization constants of the masses $Z_m = m_0/m$ and charges $Z_q = q_0/q$ turn out to be infinite.

However after the divergences are removed by means of renormalizations, some arbitrariness remains in the choice of the observed value of a parameter. This arbitrariness is eliminated by fixing the conditions of definition of observed parameter (the normalization conditions), which in fact play the role of certain boundary conditions of the problem. The arbitrariness in the choice of the values of the physical parameters thereby transforms into arbitrariness in the choice of the normalization conditions. Renormalization invariance indicates that the physical result in independent of the choice of the normalization conditions. The set of transformations from one method of normalization to another just forms the RG.

The fact that the set of transformations from one method of normalization to another forms a group was pointed out for the first time in 1953 [15], and in the following year the group properties of renormalization procedure were used to improve the results of perturbation theory in quantum electrodynamics [11].

The idea of using the RG properties to improve perturbation theory consists of the following. In the renormalized perturbation theory the property of RG invariance is a property of the whole infinite series, whereas definite terms of this series depend on the choice of the normalization condition, namely, they are not RG invariant. A change in the normalization condition corresponds to a reorganization of the whole perturbation series or some of its infinite subsequence. As a result, the information about a particular term of the perturbation series in combination with the requirement of RG invariance contains information about the infinite subsequent of the series.

To illustrate the RG invariance consider a scheme for constructing the renormalized perturbation theory. Suppose that in the theory there is some original ("bare") parameter m_0, which occurs linearly in the equation (or in the Lagrangian of a quantum mechanical system). If in the problem under consideration there is no distinguished scale with the dimension of length (modes of all scale make equal contributions to the integrals over momenta (wave numbers) of intermediate states), the corresponding integrals diverge logarithmically in the domain of large momenta. To remove the divergences, we introduce a cut over the momenta at some scale Λ. Then, a correction to the initial value of the parameter, calculated within perturbation theory, will lead to the following expression

$$m(p) = m_0 + Am_0 \ln(p/\Lambda) + \ldots \tag{5.1}$$

and after removing the cut $\Lambda \to \infty$ this correction turns out to be infinite.

To construct the renormalized perturbation theory we write the bare parameter m_0 in the form

$$m_0 = m + (m_0 - m) = m + (Z-1)m \tag{5.2}$$

where $Z = m_0/m$ is the renormalization constant. We add a term, proportional to $(Z-1)m$ (the so-called counter-term), to the part of the equation (Lagrangian) treated as a perturbation. Thus, the renormalized parameter m will occur in the unperturbed part rather than the original parameter m_0, and the perturbation series will be constructed in terms of the renormalized parameter. As a result, expression (5.1) takes the form

$$m(p) = m + Am\ln(p/\Lambda) + (Z-1)m + \ldots \tag{5.3}$$

The constant Z and m are arbitrary, which reflects the arbitrariness in splitting the equation (the Lagrangian of the system) into an unperturbed part and a perturbation.

To eliminate this arbitrariness we require that at some value of the momentum $p = \mu$ the correction to the renormalized value of the parameter m be equal to zero, i.e. we require that the following normalization condition must be satisfied

$$m(\mu) = m \tag{5.4}$$

which is achieved by the choice of the renormalization constant according to the relation

$$Z = 1 - A\ln(\mu/\Lambda) + \ldots \tag{5.5}$$

This leads to the fact that the perturbation theory series for $m(p)$ takes a form that does not contain a dependence on the cut-off momentum,

$$m(p) = m + Am\ln(p/\mu) + \ldots \tag{5.6}$$

It follows from the arbitrariness in the choice of the normalization point that when the normalization point changes $\mu \to \mu_1$ and there is a corresponding change in the renormalized value of the parameter $m \to m_1$, the result for $m(p)$ must remain unchanged, i.e.,

$$m(p) = m + Am\ln(p/\mu) + \ldots = m_1 + Am_1\ln(p/\mu_1) + \ldots \tag{5.7}$$

It can be seen that terms of the lower approximation of the renormalized perturbation theory change when the normalization conditions change. To ensure renormalization invariance the change in these terms when there is a change in the normalization point must be compensated by terms of the series corresponding to higher approximation. The requirement of renormalization invariance thereby enables one to judge the structure of the higher approximations of perturbation theory.

In 1995 it was suggested that the property of RG invariance could be used to improve the results of perturbation theory in quantum field theory [16]; this approach come to be called the renormalization group method (the RG method). If, according to this method, one finds the β-function in a lower approximation of renormalized perturbation theory and then uses this function when solving an RG differential equation like equation (1.32), the solution constructed in this way will also correspond to summing some infinite subsequence of the whole perturbation series. The examples considered in Section 4 are illustrations of this statement. In particular, if in Example 2 the next term in the perturbation series expansion (4.12) is calculated explicitly, then, from the requirement of renormalization invariance, it should have the form $g^3\,(\ln x)^2$.

This method turns out to be most effective in the case of a system containing a large number of modes with various spatial or temporal scales when interactions of the modes with nearby scales play a predominant role (a locality of interaction in scale space). In this case interaction of modes with essentially distinctive scales is achieved by means of a cascade sequence of interactions via modes of all intermediate scales. In such a system a distinctive characteristic scale does not exist and the corresponding theory possesses the property of scale invariance (scaling). Because of the absence of a distinctive scale, modes of all scales are equally significant, and this leads to logarithmically divergent expressions.

As noted by Wilson [17, 18], the existence of logarithmic divergences in the theory is a characteristic feature of a system which possesses the property of scale invariance. For example, such systems include small-scale turbulence at high Reynolds numbers [12], large-scale fluctuations in a thermodynamic system near the critical point [15], a system consisting of long polymer chains [19], the set of modes responsible for the behaviour of a dynamical system near the point of transition to chaos via an infinite sequence of bifurcations [2], and many others [21]. Renormalized perturbation theory is used to calculate an individual link is an infinite cascade chain (that specifies the form of the RG-function), and the properties of the whole chain are found by summing the contributions of separate links by solving the RG differential equation [22]. Actually, the possibility of determining the integral properties of the chain is based on using the fact that the individual links of this chain are functionally similar, i.e., they differ only in scale and the values of the numerical parameters. In the example considered at the beginning of Section 4 an analog of the transformation that describes a transition from an individual link to a neighbouring one is the operator of infinitesimal translations, the RG-function is an analog of a generator of the translation group, and the transition to the cascade chain constructed of an infinitely large number of cascade chain links can be treated as an analog of the operator of finite translations, represented in the form of an infinite sequence of infinitesimal translations.

6. ANOMALOUS DIMENSIONS, SELF-SIMILARITY OF THE SECOND KIND AND THE RENORMALIZATION-GROUP METHOD

The divergences which arise in perturbation theory are removed by a certain procedure of regularization, which reduces to the introduction of a cut-off of the integrals over wave numbers in the domain of large wave numbers in the case of ultraviolet divergences or in the domain of small wave numbers in the case of infrared divergences. A a result, a certain new parameter with a dimension of wave number is introduced into the theory, that is, an additional scale appears and due to the presence of divergences the dependence on this parameter in the limit corresponding to the removal of the regularization is non-analytical. After performing renormalizations the dependence on the scale regularizing parameter changes to a dependence on the scale parameter related to the choice of the normalization conditions. When investigating the asymptotics of the solution on the basis of dimensionality arguments it turns out that the dependence on this scale parameter does not disappear. The dependence of the

normalization constants on the choice of the normalization conditions leads to the fact that under scale transformations the renormalized parameters do not transform as they should according to primitive dimensionality arguments (without taking into account the influence of the cut-off parameter). In other words, they gain an additional (so called anomalous) dimension, which depends on the value of the coupling constant of intermode interaction [23]. It was noted in [24] that the anomalous dimension in quantum field theory is no more than an index of incomplete self-similarity (self-similarity of the second kind in the terminology of [25]) that arises in various problems of mathematical physics.

According to well-known arguments [35], the solutions corresponding to the incomplete self-similarity are asymptotics, obtained on the basis of dimensionality arguments, of a number of non-self-similar problems when some of the dimensionless parameters turn out to be very large or very small. On the basis of dimensionality arguments we can write the physical quantity Π, expressed in dimensionless form, as follows:

$$\Pi = \Phi(\Pi_0, \Pi_i) \tag{6.1}$$

where the dimensionless quantity Π_0 is related to the dimensional regularization parameter, and Π_i are other dimensionless parameters. Here we shall assume that the removal of the regularization corresponds to taking the limit as $\Pi_0 \to 0$. The presence of divergences means that the dependence of Π on the parameter Π_0 at the point $\Pi_0 = 0$ is non-analytical, that is, the finite limit

$$\lim\Pi = \Phi(0, \Pi_i) \quad \text{as} \quad \Pi_0 \to 0 \tag{6.2}$$

does not exist, but the limit

$$\lim\Pi \cdot \Pi_0^{\alpha} = \Phi_1(\Pi_i) \quad \text{as} \quad \Pi_0 \to 0 \tag{6.3}$$

exists, and the quantity Π can be written asymptotically in the form

$$\Pi = \Pi_0^{-\alpha}\Phi_1(\Pi_i) \tag{6.4}$$

where α is some positive parameter called the exponent of incomplete self-similarity. As a result, the passage from the non-self-similar problem (when the regularizing parameter Π_0 is non-zero) to their self-similar intermediate asymptotics is non-regular, the dependence on the large (small) parameter does not disappear completely, and this parameter turns out to be essential when carrying out the dimensional analysis. This case corresponds to the self-similarity of the second kind when the exponents of the power behaviour are not determined by dimensionality arguments, but are found from the solution of a certain non-linear eigenvalue problem.

In quantum field theory a calculation of the anomalous dimensions is carried out using the RG method, which enables a certain infinite subsequence of the perturbation theory series to be summed. From this the assumption naturally follows that the RG method can be used when calculating the exponents of self-similarity of the second kind. Goldenfeld et al. [24] seem to have been the first to draw attention to this fact, which demonstrated the possibility of using this method when solving the problem of non-linear diffusion (heat conduction) (the Barenblatt–Sivashinskii equation [25]) in the one-dimensional case [26]. We shall use this example to illustrate the method and to explain the somewhat abstract reasoning of this and previous sections. Unlike the approach in [26], we shall analyse the problem for a d-dimensional space [27].

7. AN EXAMPLE OF THE USE OF THE RG METHOD TO FIND THE EXPONENTS OF SELF-SIMILARITY OF THE SECOND KIND

We will analyse the equation of non-linear heat conduction, discussed in detail previously [25]

$$\{\partial_t - D[u]\Delta\}u(\mathbf{r}, t) = 0 \tag{7.1}$$

where the thermal conductivity $D[u]$ is different for heating and cooling

$$D[u] = D[1 + \epsilon H(-\partial_t u)] = \begin{cases} D & \text{when} \quad \partial_t u > 0 \\ D(1+\epsilon) & \text{when} \quad \partial_t u < 0 \end{cases} \tag{7.2}$$

and $H(x)$ is the Heaviside function. It is required to find the asymptotic form of the solution of the Cauchy problem at long times.

If we take as the initial condition

$$u(\mathbf{r}, 0) = Q_0\delta(\mathbf{r}) \tag{7.3}$$

it turns out that a solution of the form

$$u(\mathbf{r}, t) = \frac{Q_0}{(Dt)^{d/2}}\Phi\left(\frac{\mathbf{r}}{\sqrt{Dt}}, \varepsilon\right) \tag{7.4}$$

which follows from dimensionality arguments, does not exist. This is due to the fact that a solution of the form (7.4) corresponds to the conservation of the total heat content

$$q = \int u(\mathbf{r}, t)d\mathbf{r} \tag{7.5}$$

However, if we take into account the difference in the rate of heat transport during heating and cooling, the quantity q will be time-dependent. In this context problem (1.7) differs radically from the classical problem with $\varepsilon = 0$, in which $q = \text{const} = Q_0$.

To find the solution, one can use perturbation theory, regarding ε as a small expansion parameter. To do this, the perturbation, which is proportional to ε, is transferred to the right-hand side of the equation, and, using Green's function of the diffusion equation $G(\mathbf{r}, t)$ (given by relation (1.20)), we change from the differential form of the equation to the integral one that explicitly takes into account the initial condition of problem (7.4). As the result, we find

$$u(\mathbf{r}, t) = Q_0G(\mathbf{r}, t) + \epsilon D\int_0^t\int G(\mathbf{r} - \mathbf{r}', t - t')H[-\partial_t u(\mathbf{r}', t')]\Delta u(\mathbf{r}', t')d\mathbf{r}'dt' \tag{7.6}$$

The sequential iteration of Eq. (7.6) leads to a representation of the solution in the form of an infinite series in powers of ε. However, it turns out that individual terms of the perturbation series will contain integrals over t', which diverge logarithmically at the lower limit. To remove this divergence a regularization procedure is applied. This procedure consists in changing the lower limit of integration over t' by $\delta > 0$. This procedure is equivalent to the assumption that the initial distribution is specified at the instant $t = \delta > 0$ (rather then at $t = 0$) and has the form

$$u(\mathbf{r}, \delta) = Q_0G(\mathbf{r}, \delta) = \frac{Q_0}{(4\pi D\delta)^{d/2}}\exp\left(-\frac{r^2}{4D\delta}\right) \tag{7.7}$$

The initial heat content in the system $q(t = \delta) = Q_0$ and the spreading of the initial distribution over some d-dimensional volume with radius $r_0 = \sqrt{D\delta}$ corresponds to distribution (7.7). Thus, the solution will now depend on two parameters, namely, Q_0 and δ, and the dependence on the regularizing parameter δ as $\delta \to 0$ is a non-analytical (singular) one. By taking the dimensionality arguments into account we can seek a solution in the form

$$u(\mathbf{r}, t) = \frac{q(t, \varepsilon, Q_0, \delta)}{(Dt)^{d/2}}\Phi_1\left(\frac{r^2}{\sqrt{Dt}}, \frac{t}{\delta}, \varepsilon\right) \tag{7.8}$$

When searching for the asymptotic form of the solution it is assumed that the function $\Phi_1(x, y, z)$ is analytical with respect to the second independent variable as $y \to \infty$, and hence, the following limit exists

$$\lim_{t/\delta \to \infty} \frac{u(\mathbf{r}, t)(Dt)^{d/2}}{q(t, \varepsilon, Q_0, \delta)} = \Phi_1\left(\frac{r}{\sqrt{Dt}}, \infty, \varepsilon\right) = \Phi\left(\frac{r}{\sqrt{Dt}}, \varepsilon\right) \tag{7.9}$$

which corresponds to self-similarity of the second kind. The function $q(t, \varepsilon, Q_0, \delta)$ will then be asymptotically proportional to the heat content in the system at the instant t (the coefficient of proportionality can be put equal to unity owing to normalization of the function Φ).

Thus, at long times the heat content is defined by the initial parameters of the problem Q_0 and δ, which are referred to as "microscopic" one [24] (they specify the system properties at short times). These parameters are analogous to the "bare" parameters in quantum field theory.

When studying the asymptotic from of the solution we are interested in finding the low of evolution for the total heat content at long times. This corresponds to dispensing with a consideration of the initial Cauchy problem with initial distribution (7.7) in favour of considering a lightly different asymptotic problem.

The modified problem is posed as follows: for a given value of the heat content Q at the instant τ it is required to find the heat content at $t > \tau$. This means that when $t \gg \delta$

$$q(t, \varepsilon, Q_0, \delta) \to q(t, \varepsilon, Q, \tau) \tag{7.10}$$

From this it follows that now the asymptotic heat content is defined not by the initial parameters of the Cauchy problem Q_0 and δ, but by the new parameter

$$Q = Q_0 Z^{-1} \tag{7.11}$$

which should be treated as a "phenomenological" one [24]. The transfer form the "microscopic" parameter Q_0 to the "phenomenological" parameter is analogous to the renormalization procedure in quantum field theory, that is, a transfer from non-observable "bare" parameters to observable renormalized parameters.

The renormalized perturbation theory is constructed according to the scheme outlined in Section 5. In integral equation (7.6) we renormalize the parameter Q_0 by making the replacement $Q_0 \to Q$ and adding a counter-term, which compensates for this change. As a result, we obtain

$$u(\mathbf{r}, t) = QG(\mathbf{r}, t) + \varepsilon D \int_{\delta}^{t}\int G(\mathbf{r} - \mathbf{r}', t - t')H[-\partial_{t'}u(\mathbf{r}', t')]\Delta u(\mathbf{r}', t')d\mathbf{r}'dt' + Q(Z-1)G(\mathbf{r}, t) \tag{7.12}$$

Further we construct a perturbation theory series using sequential iterations, treating the first term on the right-hand side of equality (7.12) as the zeroth approximation. When splitting the right-hand side of equality (7.12) into an unperturbed part and a perturbation an arbitrariness arises related to the arbitrariness of the choice of the renormalization constant Z (and hence, of the parameter Q). The arbitrariness in the choice of the renormalization constant Z can be removed using the "normalization condition", which consists of the requirement that at some instant $t = \tau$ the heat content in the system must be equal to Q. Hence, the arbitrariness in the choice of the normalization constant Z is replaced by arbitrariness in the choice of the "normalization point" τ. The normalization condition means that at the normalization point $t = \tau$ corrections to the asymptotic solution, connected with taking the perturbations into account, have to be compensated by a contribution from the counter-term (if there are divergences, the counter-term parameters are infinite), and the value of the true solution must be identical with the value of the zeroth approximation of the *renormalized* perturbation theory (see Fig. 4, where the assumed form of the true solution for the function $q(t)$ is shown as well as both the solution $q(t) = q(\tau) = Q$, corresponding to the zeroth approximation of the renormalized perturbation theory, and that corresponding to the zeroth approximation of the unrenormalized perturbation theory $q(t) = q(\delta) = Q_0$).

The condition of renormalization invariance for the function that defines the heat content in the system has the form

$$q(t, \varepsilon, Q, \tau) = q(t, \varepsilon, Q_1, \tau_1) \tag{7.13}$$

and is identical in its structure with Eqs (1.5) and (1.14) for the examples considered above. On the basis of dimensionality arguments we can write

$$q(t, \varepsilon, Q, \tau) = Q\varphi(t/\tau, \varepsilon) \tag{7.14}$$

The function $\varphi(x, \varepsilon)$ satisfies the RG functional equation

$$\varphi(x, \varepsilon) = \varphi(\lambda, \varepsilon)\varphi(x/\lambda, \varepsilon) \tag{7.15}$$

and the boundary condition (the normalization condition)

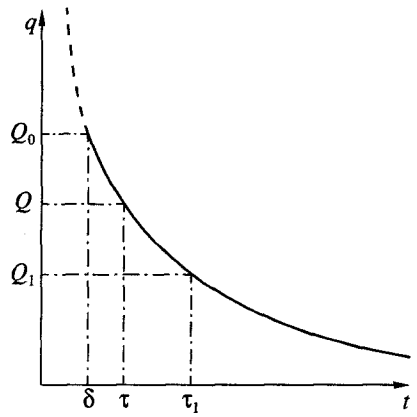

Fig. 4

$$\varphi(1, \varepsilon) = 1 \tag{7.16}$$

Changing to the RG differential equation (4.7) and its further solution, which satisfies condition (7.16), we obtain

$$\varphi(x, \varepsilon) = x^{\alpha(\varepsilon)} \tag{7.17}$$

The exponent $\alpha(\varepsilon)$ is determined by the behaviour of the function $\varphi(x, \varepsilon)$ in the vicinity of the point $x = 1$.

Following the RG method [3], let us calculate the function $\varphi(x\,\varepsilon)$ in the low-order approximation of the renormalized perturbation theory. To do this, we take the first iteration in relation (7.12) and also use expression (1.20). As the result, we obtain [27]

$$\begin{aligned} &q(t, \varepsilon, Q, \tau) = Q[1 - \varepsilon A \ln(t/\tau)], \quad Z = 1 + \varepsilon A \ln(\tau/\delta) \\ &A = (d/2e)^{d/2}/\Gamma(d/2) \end{aligned} \tag{7.18}$$

From relation (7.18) we obtain a formulae for the function $\varphi(x, \varepsilon)$ and an expression for the self-similarity exponent of the second kind

$$\alpha(\varepsilon) = -\varepsilon(d/2e)^{d/e}/\Gamma(d/2) \tag{7.19}$$

In the one-dimensional case this formula reproduces the result obtain in [26], which is in good agreement with the results of a numerical solution of the problem.

We draw attention to the fact that, when using perturbation theory, the quantity $q(t, \varepsilon, Q, \tau) \approx Q(t/\tau)^{\alpha(\varepsilon)}$ is replaced by the constant $q(\tau, \varepsilon, Q, \tau) = Q$. At first glance such an approximation seems to be unsatisfactory. However, we recall that perturbation theory is only used to calculate the behaviour of the function $q(t, \varepsilon, Q, \tau)$ near the normalization point (for calculating the RG-function). When there is "a locality of the interaction", that is, the "behaviour of a certain mode is governed by interactions with modes from the nearest vicinity (for example, in time, space, wave-number space or frequency space), this approximation turns out to be satisfactory. But it is precisely systems with local interactions that possess renormalization invariance. Similar arguments may be regarded as some justification of the RG method, which, essentially, has no rigorous mathematical basis.

8. THE ε-EXPANSION METHOD

The success in applying the RG method in problem (7.1) is due to two facts. The first is the fact that when using perturbation theory the divergences which arise can be included in the renormalization constants, which define the transition from the initial (bare, unrenormalized) parameters of the problem to phenomenological (renormalized) parameters observed in the asymptotic behaviour. In field theory this property is called renormalizability, and the corresponding theories are called renormalizable ones. The second fact is that the divergences which arise are logarithmic, that is, the normalization constants

depend logarithmically on the regularizing parameter in the limit corresponding to the removal of the regularization. As was pointed out above, the presence of logarithmic divergences denotes the locality of the non-linear interaction of the problem in space or time (the Markovran property of the process), in configuration space or in wave-number and frequency spaces (space of spatial and temporal scales).

Thus, the RG method turns out to be efficient only in the case of renormalizable theories with logarithmic divergence of the renormalization constants. This situation is encountered quite frequently, but not always. If there is no logarithmic renormalizability, one has to use more refined methods. The procedure of ε-expansion has been proposed as such a method [28]. This procedure was then successfully used in the theory of critical phenomena for calculating the scaling exponents near the critical point. It often happens that the property of renormalizability depends on the space dimension, and, whereas in the space of real dimension d the theory is non-renormalizable, in the space of another dimension $d_c \neq d$ it may turn out to be renormalizable and to contain logarithmic divergences. The problem is then considered in a space of arbitrary dimension $d = d_c + \varepsilon$, where ε is arbitrary number. While d passes through a critical value a change in the divergence character and a change in the asymptotics occur. Near the critical dimension the actual parameter of the expansion in series of the renormalized perturbation theory turns out to be proportional to ε, and this enables one to use perturbation theory even in the case when the formal expansion parameter is not small. As follows from the method of dimensional regularization of divergent integrals, developed in quantum field theory [29–32], the presence of a logarithmic divergence at $d = d_c$ leads to the fact that, in the plane of complex values of the frequency, the physical quantities calculated within the framework of perturbation theory will have a pole singularity at the point $\varepsilon = 0$. The method of ε-expansion consists of separating the contribution of this pole singularity (finding a residue at the pole) with a further analytical continuation in ε to the "physical point" $\varepsilon = d - d_c$. It has been suggested [33] that taking into account only pole singularity corresponds to separating the contribution of local intermode interactions, and the analytical continuation of the pole term indicates filtering off of the non-local interactions between modes, over which the desired solution is expanded.

In the theory of critical phenomena described by the phenomenological Ginzburg–Landau equation, for the critical dimension the value $d_c = 4$ was obtained, and the continuation in ε to the point $\varepsilon = -1$ ($d = 3$) gave good agreement with experimental results for the exponents of the power behaviour near the critical point (critical indices).

9. DIFFUSION ACCOMPANIED BY CHEMICAL REACTIONS

A more-complex problem, in which potential possibilities and technical application of the RG method can be demonstrated, is the problem of the diffusion of a material accompanied by chemical reactions [34]. The basis of a description of this problem is the equation

$$(\partial_t - D_0\Delta)C(\mathbf{r}, t) + \lambda C^{1+2\delta}(\mathbf{r}, t) = 0 \tag{9.1}$$

where λ is the reaction rate constant and $(1 + 2\delta)$ is the order of the reaction. The analysis will be carried out in a space of arbitrary dimension d.

Assume that the solution is self-similar, i.e. the functional structure of the solution does not change with time, hence, the time evolution is reduced to changing characteristic parameters of the problem, namely, the amplitude of the distribution $C(t)$ and its width $l(t)$

$$C(\mathbf{r}, t) = C(t)F(r^2/l^2(t)) \tag{9.2}$$

Note that, in the case of an initial distribution of the form $C(\mathbf{r}, 0), = q_0\delta(\mathbf{r})$, the following function corresponds to the linear problem

$$F(x^2) = \exp\{-x^2\}, \quad l^2(t) = 4D_0t, \quad C(t) = q_0/[\pi l^2(t)]^{d/2}$$

From the self-similarity hypothesis (9.2) it can be seen that, if at some instant of time the distribution was Gaussian, it will remain Gaussian. In this case the problem reduces to finding the functions $q(t)$ and $l(t)$ defined by the relations

$$q(t) = \int C(\mathbf{r}, t)d\mathbf{r}, \quad l^2(t) = 4D(t)t = \frac{d}{2}\frac{\int r^2 C(\mathbf{r}, t)d\mathbf{r}}{C(\mathbf{r}, t)d\mathbf{r}} \tag{9.3}$$

To construct the renormalized perturbation theory we will renormalize the diffusivity by making the change $D_0 \to D = Z_1^{-1}D_0$ on the left-hand side of Eq. (9.1) and adding the corresponding counter-term $(Z_1 - 1)D\Delta C$ on the right-hand side. Then, using the renormalized Green's function for the diffusion equation (where the quantity D_0 is replaced by D) we transfer to an integral equation, which contains the parameter $q_0 = \int C(\mathbf{r}, 0)d\mathbf{r}$, determined by the initial conditions of the problem. Let us renormalize this parameter by making the change $q_0 \to q = Z_2^{-1}q_0$ and adding the second counter-term $(Z_2 - 1)qG(\mathbf{r}, t)$, which compensates for this changing. The resulting integral equation will take the form

$$C(\mathbf{r}, t) = qG(\mathbf{r}, t) - \int_0^t\int G(\mathbf{r} - \mathbf{r}', t - t')[\lambda C^{1+2\delta}(\mathbf{r}', t') - (Z_1 - 1)D\Delta C(\mathbf{r}', t')](d\mathbf{r}')dt' + \\ + (Z_2 - 1)qG(\mathbf{r}, t) \tag{9.4}$$

Sequential iteration of this equation give the renormalized perturbation theory series in terms of the renormalized parameters D and q. In the first approximation, appropriate calculation gives [34]

$$C^{(1)}(\mathbf{r}, t) = \\ = q\left[G(\mathbf{r}, t) - \frac{\lambda q^{2\delta}}{D^{\delta d}}A\int G\left(\mathbf{r}, t - \frac{2\delta}{1+2\delta}t'\right)\frac{dt'}{(t')^{1-\varepsilon}} + (Z_2 - 1)G(\mathbf{r}, t) + (Z_1 - 1)Dt\Delta G(\mathbf{r}, t)\right] \tag{9.5}$$

$$\varepsilon = 1 - \delta d, \quad A = \frac{1}{(1+2\delta)^{d/2}}\frac{1}{(4\pi)^{1-\varepsilon}}$$

If we require that when $t = \tau$, the values of the functions $q(t)$ and $D(t)$ are identical with the renormalized values q and D, that is, $q(\tau) = q$ and $D(\tau) = D$, then for the renormalization constants we obtain

$$Z_1 = 1 - \frac{\lambda q^{2\delta}}{D^{\delta d}}A\frac{2\delta}{1+2\delta}\frac{1}{\tau}\int_0^\tau \frac{dt'}{(t')^{-\varepsilon}} = 1 - g\frac{2\delta}{1+2\delta}AB(1, 1+\varepsilon) \\ Z_2(\tau) = 1 + \frac{\lambda q^{2\delta}}{D^{\delta d}}A\int_0^\tau \frac{dt'}{(t')^{1-\varepsilon}} = 1 + gAB(1, \varepsilon), \quad g = \frac{\lambda q^{2\delta}}{D^{\delta d}}\tau^\varepsilon \tag{9.6}$$

here $B(\xi, \eta)$ is the beta function.

We have written the integral in Eq (9.6) in the form of the beta function due to the fact that for negative ε this integral diverges. However, it can be continued into the domain of negative ε by using the Weierstrass definition of the beta function, which holds over the whole domain of values of its argument [35]. This approach corresponds to the ε-expansion procedure and is analogous to the method of dimensional regularization of divergent expressions in quantum field theory [29–32], when the existence of divergences for a certain space dimensionality manifests itself in the pressure of pole singularities in the plane of complex dimensionalities.

Higher approximations of perturbation theory are taken into account and summation of the complete series is carried out using the RG method. To do this, we write the functions $q(t)$ and $D(t)$ in terms of dimensionless functions of dimensionless independent variables

$$q(t) = qf_2(t/\tau, g), \quad D(t) = Df_1(t/\tau, g) \tag{9.7}$$

where, by virtue of the normalization condition, we have $f_i(1, g) = 1$.

The requirement of renormalization invariance is expressed by the relations

$$qf_2(t/\tau, g) = q_1 f_2(t/\tau_1, g_1), \quad Df_1(t/\tau, g) = D_1 f_1(t/\tau_1, g_1); \quad g_1 = \frac{\lambda q_1^{2\delta}}{D_1^{\delta d}}\tau_1^\varepsilon \tag{9.8}$$

which, taking into account the normalization conditions, lead to the functional RG equations

$$f_i(t/\tau, g) = f_i(t/\tau_1, g_1), \quad i = 1, 2 \tag{9.9}$$

To solve Eqs (9.9), we introduce a new dimensionless function (the "invariant charge") [3]

$$\tilde{g}\left(\frac{t}{\tau}, g\right) = \frac{\lambda q(t)^{2\delta} t^{\varepsilon}}{D(t)^{\delta d}} \equiv g \frac{f_2^{\delta d}(t/\tau, g)}{f_1^{\delta d}(t/\tau, g)} \left(\frac{t}{\tau}\right)^{\varepsilon} \tag{9.10}$$

By virtue of Eqs (9.9) the function $\tilde{g}(x, g)$ satisfies the functional RG equation

$$\tilde{g}(x, g) = \tilde{g}\left(\frac{x}{\alpha}, \tilde{g}(\alpha, g)\right) \tag{9.11}$$

and the boundary condition $\tilde{g}(1, g) = 1$.

Using standard methods, from Eq. (9.11) we obtain the differential RG equation

$$\left\{-x\frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g}\right\}\tilde{g}(x, g) = 0$$
$$\beta(g) = \left.\frac{\partial \tilde{g}(x, g)}{\partial x}\right|_{x=1} = g[\varepsilon + 2\delta\gamma_2(g) - \delta d\gamma_1(g)], \quad \gamma_i = \left.\frac{\partial f_i(x, g)}{\partial x}\right|_{x=1} \tag{9.12}$$

Following the RG method [3], we find $\beta(g)$ in the lower approximation of renormalized perturbation theory using formula (9.6) to find $\gamma_i(g)$ [34].

Substituting the β-function into Eq. (9.12) and into the corresponding equations for $f_i(x, g)$ and then solving them we obtain

$$\tilde{g}(x, g) = \frac{gx^{\varepsilon}}{1 + (g/g^*)(x^{\varepsilon} - 1)}, \quad f_i(x, g) = \left[1 + \frac{g}{g^*}(x^{\varepsilon} - 1)\right]^{\zeta_i}$$
$$\zeta_1 = \frac{a}{\delta(1 + ad)}, \quad \zeta_2 = -\frac{1}{2\delta(1 + ad)}, \quad g^* = \frac{\varepsilon}{2\delta a(1 + ad)}, \quad a = \frac{\delta}{1 + 2\delta} \cdot \frac{1 - \delta d}{2 - \delta d} \tag{9.13}$$

Note that a certain critical dimension $d_c = 1/\delta = 2/(n - 1)$ arises in the theory (corresponding to $\varepsilon = 0$), which depends on the order of the chemical reaction, such that above and below this dimension the regimes of asymptotical behaviour of the system turn out to be different. Analysis of these solutions for different signs of λ (the case $\lambda > 0$ corresponds to the reaction of absorption, whereas $\lambda < 0$ corresponds to the reaction of the formation of a substance) and various values of the space dimension enables one to find possible regimes of system behaviour [34]. In particular, if the space dimension is below the critical one, in the case of a reaction of the formation of a substance the RG method predicts the occurrence of "blow-up regimes" when the process of localization of the initially smeared-out distribution develops; such regimes are known from the results of a numerical analysis of the corresponding problem [36].

10. WILSON'S FORMULATION OF THE RENORMALIZATION GROUP METHOD

A somewhat different and clearer approach to understanding the ideas and use of the RG method has been formulated by Wilson [17, 18]. According to Wilson, the RG method is a method of investigating a multimode system with many characteristic spatial and temporal scales. The absence of a distinguished characteristical scale (modes of all scales are equally important) is related to the fact that there is a locality of interactions in scale space, i.e. interaction only occurs between modes of close scales, whereas the interaction between modes with fairly different scales occurs by a cascade sequence of interactions via modes of intermediate scales. In section 2 such processes were called Markovran process in a generalized sense. The presence of the cascade mechanism when there is no distinguished characteristic scale leads to identity (apart from a scale transformation) of the pattern of fluctuations of dynamic modes of different scales (self-similarity, scale invariance and scaling).

The RG method in Wilson's formulation reduces to changing from a real multimode system to a certain equivalent system with a smaller number of modes but having the same behaviour in the domain

of large-scale and slow processes. The equations for large-scale (slow) modes are obtained by averaging over scales (periods) of small-scale (fast) modes. This approach is not particularly new, and is well-known and quite widely used. In particular, averaging over the degrees of freedom of molecular motions which reduces to the introduction of the empirical molecular transport coefficients, corresponds to the hydrodynamic description of a many particle system used in statistical physics. In the theory of non-linear oscillations the Krylov–Bogolyubov method is successfully used, in which the evolutionary equations for slow variables, namely, the amplitudes and phase shifts, are obtained after averaging over fast variables that correspond to oscillations at the fundamental frequency [37]. In both cases pointed above the spectra of slow and fast modes are quite strongly separated, and the averaging procedure over the fast modes is well-defined. However, in the case of continuous spectra there is no sharply defined boundary between the slow and fast modes, and the procedure of reducing the number of modes is carried out using a somewhat more complicated scheme by means of a sequence of partial averaging over narrow spectral bands with ever increasing periods (scales).

This procedure is carried out as follow [4, 5]. Suppose the system contains modes with a spectrum having an upper limit by some value of the wave number Λ (or the frequency Ω). We split the spectrum into two parts: $0 \leqslant k \leqslant \Lambda e^{-\tau}$ and $\Lambda e^{-\tau} \leqslant k \leqslant \Lambda$, where τ is some positive parameter. By convention we will call modes from the first wave-number interval slow modes, and those from the second interval will be called fast modes (in turbulence theory, according to the terminology of A. M. Obukhov, these modes are called macro- and microcomponents respectively [12]). If the value of τ is sufficiently small, the number of fast modes will not be large. In non-linear system the equations for the fast and slow modes are not separated. However, in the equation for the fast modes we can regard the functions corresponding to slow modes as constants; this is similar to the approximation of Wentzel–Kramers–Brillouin (λ) (the WKB approximation), which is well known in quantum mechanics. After solving the equations for the fast modes when slow modes are present we substitute these solutions into the equations for the slow modes and average over the periods of the fast modes. As a result, we obtain the equations for the slow modes taking account of the averaged influence of the fast modes.

In the case of the lattice model the above method of eliminating the fast modes corresponds to changing from considering individual elementary cells to blocks, containing several cells, and a description of the system by specifying averaged values in a block ("large-grain coarsening"). This change is referred to as a Kadanoff transformation [38], and the sequential performance of Kadanoff transformations reduces to changing to blocks of ever increasing size and decreasing their total number. In the theory of polymer systems [19] the clustering of a certain number of sequential monomers into subunits and the calculation of the parameters specifying the sizes and coupling constants of the subunits using information on the sizes and coupling constants of the monomers corresponds to the Kadanoff transformation.

After eliminating the fast modes the spectrum of the remaining modes will be bounded by the domain $0 \leqslant k \leqslant \Lambda e^{-\tau}$. We perform the scale transformation $k \to k' = ke^{\tau}$, as a result of which the range of the wave-number spectrum takes the previous value $0 \leqslant k' \leqslant \Lambda$. As a result of partial averaging and scale transformation the parameters in the equation for the slow modes change (renormalize), but the form of the equation remains the same (though maybe only approximately) if the law for transforming the amplitudes in the scale transformation is suitably chosen. Thus, the operation of RG transformation is reduced to a combination of partial averaging over a narrow bond of the fast mode spectrum (the Kandanoff transformation) and a scale transformation.

Suppose the system is specified by a set of numerical parameters $\{\mu^{(0)}\}$. For example, these could be the parameters of the Hamiltonian of the system (the masses and coupling constants), which define the (distribution density over the states $P \sim \exp\{-H/kT\}$. If the subject of interest is the system behaviour in the large-scale domain, a knowledge of the distribution function over all states is unnecessary, and the distribution over large-scale states can be found by averaging over small-scale states, i.e., by eliminating the small-scale states. The basis of RG approach is the assumption that the structure of the effective Hamiltonian, which describes the distribution over large-scale states, remains the same, but with the changed (renormalized) values $\{\mu\}$ replacing the initial parameters of the problem $\{\mu^{(0)}\}$.

If the set of original parameters $\mu_i^{(0)}$, which specify the system, is represented by a point in the space of parameters $\{\mu^{(0)}\}$, the operation of RG transformation can be represented symbolically in the form

$$\{\mu^{(0)}\} \to \{\mu(\tau)\} = \hat{R}(\tau)\{\mu^{(0)}\} \tag{10.1}$$

where $\hat{R}(\tau)$ is the RG transformation operator acting in the space of parameters $\{\mu\}$. Successive performance of the operations of RG transformation corresponds to motion of the representative point

in the space of parameters, where τ plays the role of time and the vector $\{\mu^{(0)}\}$ corresponds to the initial conditions. In this sense the operator $\hat{R}(\tau)$ corresponds to the change in the initial conditions under a shift of the initial instant of time, and the requirement of RG invariance means the system behaviour does not depend on the method of specifying the initial conditions, similar to the case described in Section 1, Example 1.

Here it is assumed that (at least asymptotically) the following relation holds

$$\{\mu(\tau+\tau')\} = \hat{R}(\tau+\tau')\{\mu^{(0)}\} = \hat{R}(\tau')\hat{R}(\tau)\{\mu^{(0)}\} = \hat{R}(\tau')\{\mu(\tau)\} \tag{10.2}$$

It can be seen that the operator $\hat{R}(\tau)$ possesses the group property

$$\hat{R}(\tau+\tau') = \hat{R}(\tau')\hat{R}(\tau), \quad \hat{R}(0) = \hat{I} \tag{10.3}$$

However, since the process of averaging over fast modes is irreversible ($\tau', \tau \geqslant 0$), there is no inverse element, and therefore, the set of transformation $\hat{R}(\tau)$ is a semigroup rather than a group.

The change to an equation for the modes of ever larger scales corresponds to an increase in the parameter. When $\tau \to \infty$ the representative point tends to some limiting value $\{\mu^*\}$, which satisfies the condition

$$\hat{R}(\tau)\{\mu^*\} = \{\mu^*\} \tag{10.4}$$

From condition (10.4) it follows that $\{\mu^*\}$ is a fixed (stationary) point of the RG transformation. Thus, the asymptotic behaviour in the large-scale domain $\tau \to \infty$ is specified by the set of parameters μ_i^*, found from Eq. (10.4). In other words, the set of parameters $\{\mu^*\}$ is an eigenvector of the RG transformation operator. These parameters are specified by the properties of the RG transformation and do not depend on the choice of the original microscopic parameters of the system $\mu_i^{(0)}$, and this corresponds to university of the system asymptotic behaviour.

Functional equation (10.2) can be written in the form of a differential equation

$$\frac{\partial}{\partial\tau}\{\mu(\tau)\} = \hat{H}(\{\mu(\tau)\}\{\mu(\tau)\}), \quad \hat{H}(\{\mu(\tau)\}) = \frac{\partial\hat{R}(\tau)}{\partial\tau} \tag{10.5}$$

where, in the case when the group composition law (10.2) is satisfied, the matrix operator $\hat{H}$, defined by the relation $\hat{R}(\delta\tau) = \hat{I} + \hat{H}\delta\tau$, turns out to be independent of τ and will be a generator of the group of RG transformations. In this case, according to Eq. (10.2), the operator of finite RG transformations can be represented in the form

$$\hat{R}(\tau) = \exp(\hat{H}\tau) \tag{10.6}$$

similar to formula (4.5) for a group of operators of the shifts of the function argument.

The equation for finding the fixed points is formulated as

$$\hat{H}(\{\mu^*\})\{\mu^*\} = 0 \tag{10.7}$$

Like the trivial examples of RG invariance considered in Section 1, in this case the asymptotic form of the large-scale behaviour is independent of the method of specifying the "initial conditions" $\{\mu(\tau)\}|_{\tau=0} = \{\mu^{(0)}\}$.

When considering critical phenomena all physical systems, whose initial mapping points $\{\mu^{(0)}\}$ under RG transformations $\hat{R}(\tau)|_{\tau\to\infty}$ tend to a single, fixed point, possess the same distribution functions in the large-scale domain. A knowledge of the eigenvectors of problem (10.7) enables one to investigate the behaviour of the representative point of the system $\{\mu(\tau)\} = \{\mu^* + \delta\mu(\tau)\}$near the fixed point. For this, it is sufficient to find, within the framework of perturbation theory, the corrections to the eigenfunctions and eigenvalues using the expression for the operator $\hat{H}$ linearized near the fixed point and representing a solution of the perturbed system in the form of an expansion in eigenfunctions o the unperturbed problem. In this way one can explain the universality of the large-scale behaviour of various systems and, in particular, the universality of so called "critical indices" (critical exponents) that specify the behaviour near the critical temperature, which turns out to be independent of the initial parameters of the problem $\{\mu^{(0)}\}$. In similar manner one can explain the universality of the behaviour of various non-linear systems when there is a transition to chaos via a sequence of doublings of the period [20].

11. THE RENORMALIZATION-GROUP METHOD IN TURBULENCE THEORY

The regime of fully developed turbulence in the inertial spectral range, which occurs at large Reynolds numbers, is a typical example of a multi mode system, for which the RG method should work successfully. This is related to the property of locality of the intermode interactions in wave-number space and the cascade mechanism of interactions between modes with essentially different wave numbers (scales). In the case of a hydrodynamic system the nature of the locality can be easily explained: the interaction between eddies with essentially different scales is reduced to the kinematic effect of the transport of small-scale eddies by large-scale eddies without appreciable distortion of their form, that is, without energy redistribution between modes. Interactions between modes of different scales form a cascade chain, consisting of functionally similar links (i.e. distinguished by their scales and the set of numerical parameters). Due to the locality of the interactions, in order to find the properties of an individual chain link it is sufficient to take into account the interaction of a given mode solely with modes which are neighbours in wave-number space. In turn, information about the characteristics of an individual link of the chain (specified by the β-function) enables one to find the properties of a long cascade chain by solving the RG differential equation that follows from the renormalization invariance. A review of the application of the RG method for describing turbulence can be found in [12, 13].

As a specific example we will consider a calculation of the turbulence viscosity $\tilde{\nu}(k)$, which phenomenologically takes into account processes of mode decay for modes with wave number k as a result of non-linear interactions with all the remaining modes [12, 40, 41]. The turbulent viscosity differs from the subgrid viscosity discussed in Section 3, which takes into account the averaged effect of interactions with only subgrid modes.

As the mathematical model of turbulence a system of Navier–Stokes equations is used with the presence of an external random force of the Gaussian white-noise type. It is assumed that a statistical solution of the hydrodynamic equations must reproduce the results of Kolmogorov's phenomenological theory [12]. The external random force $f(k, \omega)$ is specified by a covariance of the form

$$\langle f_i(\mathbf{k}, \omega) f_i(\mathbf{k}', \omega') \rangle = \delta_{ij} D(k) (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') (2\pi) \delta(\omega + \omega') \tag{11.1}$$

(d is the space dimensionality).

The similarity of the physical pattern in different scales suggests that the function $D(k)$ must have a power form and can be written as

$$D(k) = 2D_0 k^{-y} \tag{11.2}$$

If $y = d$, the dimension of the constant D_0 is identical with the dimension of the rate of energy dissipation, which, according to Kolmogorov's theory, is the only essential dimensional parameter that characterizes the properties of the turbulent fluid in the inertial range of the spectrum. For this reason, the value $y = d$ should be regarded as corresponding to the real (physical) theory. However, when $y = d$ divergences (infrared) arise in the theory, which cannot be removed by renormalization (non-renormalizability occurs). An analysis of the degrees of divergence shows that when $y = d - 4 = y_c$ the theory will contain logarithmic divergences in the ultraviolet domain, which can be included in the normalization constant of the viscosity (renormalizability occurs).

Following the procedure of ε-expansion, we shall construct the theory for the case when $y = y_c + \varepsilon$ in the vicinity of the point $\varepsilon = 0$ with subsequent analytical continuation to the physical point $\varepsilon = 4$, as is done when applying the dimensional regularization method in quantum field theory [29, 30]. In this case D_0 no longer has the dimension of the rate of energy dissipation. However, it is possible to keep the dimension of D_0 unchanged by putting $D(k) = D_0 k^{-d} (k/\mu)^{4-\varepsilon}$ [41], where μ is some parameter with the dimension of wave number.

The effective viscosity $\tilde{\nu}(k, \omega)$ can be defined in terms of inverse Green's function by the relation [12]

$$G^{-1}(k, \omega) = -i\omega + \nu_0 k^2 - \Sigma(k, \omega) = [G^{(0)}(k, \omega)]^{-1} - \Sigma(k, \omega) = -i\omega + \tilde{\nu}(k, \omega) k^2 \tag{11.3}$$

where ν_0 is the initial (bare) viscosity and $\Sigma(k, \omega)$ is a quantity, which, in quantum field theory, is called the self-energy operator.

In the non-renormalized perturbation theory the following is used as the zeroth approximation for Green's function

$$G^{(0)}(k, \omega) = [-i\omega + \nu_0 k^2]^{-1} \tag{11.4}$$

Renormalization of the viscosity reduces to making the change $\nu_0 \to \nu = \nu_0 Z^{-1}$ in Eq. (11.3) and adding a counter-term to the self-energy operator, which compensates for this change,

$$\Sigma(k, \omega) \to \Sigma^R(k, \omega) = \Sigma(k, \omega) + (\nu - \nu_0)k^2 = \Sigma(k, \omega) + \nu(1 - Z)k^2 \tag{11.5}$$

In the renormalized perturbation theory the following is used as the zeroth approximation

$$G^{(0)R}(k, \omega) = [-i\omega + \nu k^2]^{-1} \tag{11.6}$$

In what follows we shall consider only the static effective viscosity

$$\tilde{\nu}(k) = \nu_0 - \Sigma(k, 0)/k^2 = \nu - \Sigma^R(k, 0)/k^2 \tag{11.7}$$

Renormalization invariance implies arbitrariness in splitting the right-hand side of Eq. (11.7) into an unperturbed part and a perturbation. This arbitrariness finds reflection in the choice of the renormalization constant Z and can be reduced to arbitrariness in the choice of the normalization point, i.e., the value of the wave number μ at which the effective viscosity will be the same as the renormalized viscosity ν:

$$\tilde{\nu}(\mu) = \nu, \quad \text{or} \quad \Sigma^R(\mu, 0) = 0 \tag{11.8}$$

Normalization condition (11.8) performs the function of the boundary conditions, and the RG invariance corresponds to the presence of arbitrariness in the method of specifying these conditions.

The effective viscosity $\tilde{\nu}(k)$ is function of k, D_0 and the two parameters ν and μ, which specify boundary conditions (10.8). Due to the fact that the form of the solution is independent of the method of specifying the boundary conditions, we have

$$\tilde{\nu} = f(k, D_0; \nu, \mu) = f(k, D_0; \nu_1, \mu_1) \tag{11.9}$$

On the basis of the dimensionality arguments and relation (11.9) we can write

$$f(k, D_0; \nu, \mu) = \nu\varphi(k/\mu, D_0/(\nu^3\mu^\varepsilon)) = \nu_1\varphi(k/\mu_1, D_0/\nu_1^3\mu_1^\varepsilon) \tag{11.10}$$

It can easily be shown that the function

$$\tilde{g}(x, g) = gx^{-\varepsilon}/\varphi^3(x, g), \quad (g = D_0/\nu^3\mu^\varepsilon, x = k/\mu) \tag{11.11}$$

is invariant under the RG transformation $\nu \to \nu_1$, $\mu \to \mu_1$, i.e.,

$$\tilde{g}(k/\mu, D_0/\nu^3\mu^\varepsilon) = \tilde{g}(k/\mu_1, D_0/\nu_1^2\mu_1^\varepsilon) \tag{11.12}$$

and satisfies the normalization condition

$$\tilde{g}(1, g) = g \tag{11.13}$$

The function $\tilde{g}(x, g)$ is the actual parameter of the series expansion of the renormalized perturbation theory, which depends on the wave number. It is the analog of the invariant charge in quantum field theory [3]. From the condition that the function $\tilde{g}$ is RG invariant it follows that it satisfies functional RG equation (1.28) and differential RG equation (1.31).

In the lowest approximation of renormalized perturbation theory we obtain for the function $\varphi(x, g)$ [40]

$$\varphi(x, g) = 1 + A(d, \varepsilon)q[x^{-\varepsilon} - 1]/\varepsilon \tag{11.14}$$

(the last term in the square brackets takes into account the contribution of the counter-term, which ensures that the normalization conditions is satisfied). From condition (11.3) it can be seen that the function φ possesses a pole singularity at the point $\varepsilon = 0$, which reflects the presence of logarithmic divergence in the ultraviolet region of wave number spectrum. In accordance with the procedure of ε-expansion, we retain only the contribution of this pole singularity by putting by putting $\varepsilon = 0$ in the expression for $A(d, \varepsilon)$ (the residue at the point $\varepsilon = 0$). Appropriate calculation gives [33, 40]

$$A(d, 0) = \frac{d-1}{4(d+2)} \frac{s_d}{(2\pi)^d}, \quad s_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{11.15}$$

Using relations (11.11) and (11.14) and the definition of the β-function we obtain

$$\beta(g) = -\varepsilon g + 3A(d, 0)g^2 = g[-\varepsilon + 3A(d, 0)g] \tag{11.16}$$

From equality (11.16) it follows that two fixed points exist, namely, the trivial one $g^* = 0$, which corresponds to the absence of interactions (asymptotic freedom), and a non-trivial one $g^* = \varepsilon/(3A(d, 0))$. Since the derivative of the β-function at the non-trivial fixed point is positive, this point will be stable in the infrared limit as $x = k/\mu \to 0$ (the large-scale domain). Thus, in the asymptotic domain, which corresponds to the inertial interval, we obtain the following expression for the effective viscosity

$$\tilde{\nu}(k) \approx (D_0/q^*)^{1/3} k^{-\varepsilon/3} = [3A(d, 0)/\varepsilon]^{1/3} D_0^{1/3} k^{-\varepsilon/3} \equiv \nu_T(k) \tag{11.17}$$

When $\varepsilon = 4$, formula (11.17) reproduces the Richardson–Kolmogorov law for turbulent viscosity [12], which follows from dimensionality arguments. However, the RG method enables one to find the numerical coefficient if one knows the relation between the rate of energy dissipation $\mathscr{E}$ and the parameter D_0 in (10.2), and also enables one to calculate theoretically the universal Kolmogorov constant. A method of finding this relation has been proposal [12], as a result of which the following expression has been obtained for the Kolmogorov constant

$$C_K = (2/3)[2(d+2)]^{1/3} (=1.44 \quad \text{for} \quad d = 3) \tag{11.18}$$

which is in good agreement with experimental data.

We draw attention to the fact that, by using the RG method, one can find an expression for the effective viscosity, which holds not only in the asymptotic region as $k \to 0$, but also in a wider region of the wave-number spectrum. In particular, if we assume that, in the small-scale region $k \to \infty$ the value of the effective viscosity tends to the molecular one $\tilde{\nu}(k) \to \nu_0$, solution of the RG differential equation gives

$$\tilde{\nu}(k) = [\nu_0^3 + \nu_T^3(k)]^{1/3} \tag{11.19}$$

The function $\nu_T(k)$ is defined by formula (11.17).

Within the framework of Wilson's formulation of the RG method the corresponding analysis is known as the Yakhot–Orszag turbulence theory [43] (for more detail see [12] and the review [39]).

12. CONCLUSION

We shall not list the numerous applications of the RG method in various branches of mechanics and applied mathematics since this is difficult to do, the aim of this review being to outline the idea and illustrate some technical approaches, which are the basis of the method. Nevertheless, one more possibility of applying the RG method should be mentioned relating to the construction of asymptotic solutions of differential equations with a small parameter ε, within the framework of the perturbation-theory method. We are referring to so-called singular perturbations when the search for a solution in the form of a direct series expansion in powers of the small parameter ("naive" perturbation theory) does not lead to uniformly suitable expressions [44] due to the appearance of so-called secular terms in the series expansion. Such a situation arises when the equation contains a small parameter with the highest derivative or when a symmetry group of the unperturbed differential equation is broken by a perturbation (for example, the type of differential equation changes).

The traditional treatment of similar problems within the framework of the method of stretched co-ordinates or parameters, matching of the outer and inner asymptotic expansions, the Krylov–Bogolyubov

averaging procedure, the method of multiple scales, and other approaches are most commonly based on an analysis of the physical pattern of the phenomena described. Such as analysis also governs the choice of the method of investigation. In this context, the RG method has the advantage that it provides a unified approach to the description of phenomena, which enables one to distinguish the stable structural properties of the system against the background of insignificant details [45]. The starting point of this approach is simple perturbation theory, whose application does not require a priori knowledge of the physical pattern of the phenomena. The further improvement of perturbation theory based on the property of renormalization invariance of the whole series "automatically" chooses the method of investigation, which is adequate to the problem under consideration. Suitable examples can be found in [46].

However, the RG method not only enables one to improve perturbation theory, but also enables one to simplify the asymptotic analysis of the singular behaviour of the solutions, which, in the vicinity of a singular point, becomes scale invariant (self-similar), and the technique of RG analysis enables one to calculate corrections to the exponents of the power behaviour of the solution that follow from "naive" dimensionality arguments (the exponents of incomplete self-similarity [25] or anomalous dimensions).

It is worth drawing attention to following two points.

The successes achieved with the RG method, in particular, in describing of the universal behaviour of a thermodynamic system near the critical point in phase transitions of the second kind, gives season to hope that this method can be widely used in the analysis of complex systems which traditional approaches are unable to describe. Basically, these hopes are connected with Wilson's formulation of the RG method, which, unlike the quantum field formulation, is clearer and more accessible to wide variety of researchers who are not very familiar with the quantum field theory formalism. As has been pointed out, Wilson's formulation is based on the idea of the sequential reduction of the number of modes in a multimode system by means of Kadanoff's procedure of averaging over a narrow band of the spectrum of fast (small-scale) modes ("large-grain coarsening"). Kadanoff's procedure has often come to be identified with the RG method and this procedure began formally to be used without a preliminary analysis of the problem of the existence of RG invariance. As an example, when calculating the turbulent viscosity in the Yakhot–Orszag theory [43] or when finding the effective transport coefficients in a random velocity field or in a randomly heterogeneous medium [48, 49] the transport coefficient $\tilde{K}(q, K_0)$ was considered, which governs the diffusion processes and depends on the wave number q and the molecular transport coefficient K_0. When introducing a cut-off of the wave numbers it was assumed that the averaged effect of fast modes, removed in the cut-off, with wave numbers $q > \Lambda$ can be phenomenologically taken into account by replacing the molecular transport coefficient K_0 by the renormalized value $K(\Lambda)$, and the transport coefficient $\tilde{K}(q, \Lambda, K(\Lambda))$, which depends on the wave number q, the cut-off parameter Λ and the renormalized transport coefficient $K(\Lambda)$ was considered. Further, in the lower approximation of perturbation theory (renormalized) the coefficient $\tilde{K}(q, \Lambda, K(\Lambda))$ was found in the large-scale limit as $q \to 0$, and the quantity

$$\lim_{\delta\Lambda \to 0} \frac{\tilde{K}(0, \Lambda, K(\Lambda)) - \tilde{K}(0, \Lambda - \delta\Lambda, K(\Lambda))}{\delta\Lambda} = F(K(\Lambda))$$

was calculated which was identified with the quantity $dK(\Lambda)/d\Lambda$. By integrating the differential equation $dK/d\Lambda = F(K)$ with the boundary condition $K(\infty) = K_0$ the function $K(\Lambda, K_0)$ was found, which was then identified with the function $\tilde{K}(q, K_0)$. The procedure when calculations are carried out in the limit as $q/\Lambda \to 0$ and it is then assumed that $\Lambda = q$ seems to be too contradictory and cannot be justified. In addition, the procedure described is not based on the RG transformation, since the Kadanoff procedure is not supplemented with a scale transformation, being a constituent part of the RG transformation.

The second point, to which we must call attention, relates to the treatment of the role of local and non-local interactions within the framework of the RG approach. The technique for finding the equation for $K(\Lambda)$ described above was based on considering the essentially non-local direct effect of fast modes the wave-number range $\Lambda - \delta\Lambda < k < \Lambda$ on slow modes with wave numbers q as $q/k \to 0$. As it applies to turbulence theory, it was suggested in [50–52] that for small ε local interactions are small, and this proposal has received the name of the "distant interaction principle" [53]. The statement that, using the RG method, non-local interactions are taken into account and the role of local interactions is small forms the basis of the technique for making calculations in Yakhot–Orszag RG turbulence theory [43]. But this contradicts Wilson's viewpoint of the RG method as a way of describing local interactions, which are responsible for cascade processes and whose presence is revealed in the occurrence of logarithmic divergences and singularities in ε near the "logarithmic theory" [17, 18]. The assertion has been made [39, 54] that the criterion of the significance of local and non-local interactions should be

their relative contribution not to the physical quantities under consideration, but to the β-function, which, within the framework of RG description, contains all information on the system. Estimates [39, 54] have shown that, according to this criterion, near the"logarithmic" theory local interactions play a predominant role, which agrees with Wilson's viewpoints.

Thus, despite the impressive achievements of the RG method, its application requires some accuracy. Firstly, one must not identify the RG method solely with Kadanoff's procedure for reducing the number of modes in a multimode system by sequential averaging over a band of a fast-mode spectrum (unfortunately, this confusions seems to be widespread). Secondly, one must take into account that RG invariance does not always occur, but only in "logarithmic" theories (which contain logarithmic divergences). If the theory is not "logarithmic", one must apply, by analogy with the dimensional regularization method in quantum field theory, the procedure of ε-expansion, that is, carry out the analysis close to "logarithmic" theory (which corresponds to $\varepsilon = 0$) and than perform analytical continuation in ε to the point that corresponds to the real theory (see, for example [55]. It is more convenient to use this approach within the framework of the field-theory formulation of the RG method outlined above, not containing the additional (and in some cases contradictory) assumptions sometimes used in Wilson's formulation of the method.

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